Towards an Inverse Scattering theory for non decaying potentials of the heat equation§

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Abstract. The resolvent approach is applied to the spectral analysis of the heat equation with non decaying potentials. The special case of potentials with spectral data obtained by a rational similarity transformation of the spectral data of a generic decaying potential is considered. It is shown that these potentials describe N solitons superimposed by Bäcklund transformations to a generic background. Dressing operators and Jost solutions are constructed by solving a $\overline{\partial}$-problem explicitly in terms of the corresponding objects associated to the original potential. Regularity conditions of the potential in the cases $N = 1$ and $N = 2$ are investigated in details. The singularities of the resolvent for the case $N = 1$ are studied, opening the way to a correct definition of the spectral data for a generically perturbed soliton.

1. Introduction

The operator

$$\mathcal{L}(x, \partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x), \qquad x = (x_1, x_2) \tag{1.1}$$

which defines the well-known equation of heat conduction, or heat equation for short, from the beginning of the seventies [1, 2] is known to be associated to the Kadomtsev–Petviashvili equation in its version called KPII

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2}. \tag{1.2}$$

The spectral theory for the equation of heat conduction with real potential $u(x)$ was developed in [3]–[6], but only the case of potentials rapidly decaying at large distances in the x-plane was considered. We are interested in including in the theory potentials with one-dimensional asymptotic behaviour and in particular potentials describing N solitons on a generic background. In trying to solve the analogous problem for the nonstationary Schrödinger operator, associated to the KPI equation, a new general approach to the inverse scattering theory was introduced, which was called resolvent approach. See [7]–[15] for the nonstationary Schrödinger operator and [16] for the Klein–Gordon operator. Some results for the operator (1.1) were given in [17]. For the specific case of N solitons on a background for the KPI equation see especially [15, 18].

Here we apply the same approach to the heat equation and construct a potential $u'(x)$ describing N solitons superimposed to a generic background potential $u(x)$.

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Superimposition is performed by means of a rational similarity transformation of the spectral data of $u(x)$. This procedure supplies us not only with the Bäcklund transformation of the potential $u(x)$, but also with the corresponding Darboux transformations of the Jost solutions and with the spectral theory for the transformed potential $u'(x)$. All related mathematical entities such as the extended resolvent M', the dressing and dual dressing operators ν' and ω' and the Jost and dual Jost solutions Φ' and Ψ' corresponding to $u'(x)$ are given explicitly in terms of the same objects associated to the background potential $u(x)$. In [19, 20] some preliminary results were presented for $N = 1$ and 2 by using recursively the binary Darboux transformations.

We show that the main mathematical object of the theory, i.e. the extended resolvent M', is given as a sum of two terms. The first one is obtained by dressing with the operators ν' and ω' the resolvent M_0 of the bare heat operator $\mathcal{L}_0(x, \partial_x) = -\partial_{x_2} + \partial_{x_1}^2$, while the second one, m', takes into account the discrete part of the spectrum. The dressing operators ν' and ω' are constructed by solving a $\overline{\partial}$-problem for the transformed spectral data.

The theory with respect to the nonstationary Schrödinger equation is in some respects simpler and in some other respects unexpectedly more difficult. We give the explicit expression for Φ', Ψ', and u' for any N, but the reality and regularity conditions for the potential are rather involved and we examine in details only the cases $N = 1, 2$. We study the singularities of the resolvent in the case $N = 1$. This resolvent can be used for investigation of the spectral theory of operator (1.1) with the potential being a perturbation of the potential $u'(x)$ obtained by adding to it a 'small' function $u_2(x)$ rapidly decaying on the x-plane. It can be shown that under such a perturbation of the potential the Jost solutions get singularities more complicated than poles, but on the other side they have no additional cuts on the complex plane of spectral parameter, in contrast with the nonstationary Schrödinger case. This means, however, that also in the case of the perturbed heat equation the standard definition of spectral data for a generic non decaying potential must be modified. The solution of this problem is deferred to a future work.

2. Direct and inverse problems in the case of rapidly decaying potentials

In the framework of the resolvent approach we work in the space $\mathcal{S}'$ of tempered distributions $A(x, x'; q)$ of the six real variables x, x', $q \in \mathbb{R}^2$. It is convenient to consider q as the imaginary part of a two-dimensional complex variable $\mathbf{q} = \mathbf{q}_\Re + \mathrm{i}\mathbf{q}_\Im = (\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{C}^2$ and to introduce the "shifted" Fourier transform

$$A(p; \mathbf{q}) = \frac{1}{(2\pi)^2} \int \mathrm{d}x \int \mathrm{d}x' \, \mathrm{e}^{\mathrm{i}(p+\mathbf{q}_\Re)x - \mathrm{i}\mathbf{q}_\Re x'} A(x, x'; \mathbf{q}_\Im) \tag{2.1}$$

where $p \in \mathbb{R}^2$, $px = p_1 x_1 + p_2 x_2$ and $\mathbf{q}_\Re x = \mathbf{q}_{1\Re} x_1 + \mathbf{q}_{2\Re} x_2$. We consider the distributions $A(x, x'; q)$ and $A(p; \mathbf{q})$ as kernels in two different representations, the x-representation and the $(p, \mathbf{q})$-representation, respectively, of the operator $A(q)$ (A for short). The composition law in the x-representation is defined in the standard way, that is

$$(AB)(x, x'; q) = \int \mathrm{d}x'' \, A(x, x''; q) \, B(x'', x'; q). \tag{2.2}$$

Since the kernels are distributions this composition is neither necessarily defined for all pairs of operators nor associative. In terms of the $(p, \mathbf{q})$-representation (2.1) this

composition law is given by a sort of a "shifted" convolution,

$$(AB)(p;\mathbf{q}) = \int \mathrm{d}p' A(p-p';\mathbf{q}+p')B(p';\mathbf{q}). \tag{2.3}$$

On the space of these operators we define the conjugation A^* and the shift $A^{(\mathbf{s})}$ for the complex parameter $\mathbf{s}\in\mathbb{C}^2$, that in the x- and $(p,\mathbf{q})$-representations read, respectively,

$$A^*(x,x';q) = \overline{A(x,x';q)}, \qquad A^*(p;\mathbf{q}) = \overline{A(-p;-\overline{\mathbf{q}})} \tag{2.4}$$

where bar denotes complex conjugation, and

$$A^{(\mathbf{s})}(p;\mathbf{q}) = A(p;\mathbf{q}+\mathbf{s}), \qquad A^{(\mathbf{s})}(x,x';q) = \mathrm{e}^{\mathrm{i}\mathbf{s}_{\Re}(x-x')}A(x,x';q+\mathbf{s}_{\Im}). \tag{2.5}$$

For any operator A, thanks to the fact that its kernel belongs to the space $\mathcal{S}'(\mathbb{R}^6)$, we can consider its $\bar\partial_j$-differentiation (in the sense of distributions) with respect to the complex variables $\mathbf{q}_j$ $(j=1,2)$:

$$(\bar\partial_j A)(p;\mathbf{q}) = \frac{\partial A(p;\mathbf{q})}{\partial\overline{\mathbf{q}}_j}, \qquad (\bar\partial_j A)(x,x';q) = \frac{\mathrm{i}}{2}\Big(x_j - x'_j + \frac{\partial}{\partial q_j}\Big)A(x,x';q), \tag{2.6}$$

where the formula for the kernel of $\bar\partial_j A$ in the x-representation is obtained by (2.1).

The set of differential operators is embedded in the introduced space of operators by means of the following extension procedure. Any given differential operator $\mathcal{A}(x,\partial_x)$ with kernel

$$A(x,x') = \mathcal{A}(x,\partial_x)\delta(x-x'), \tag{2.7}$$

$\delta(x-x') = \delta(x_1-x'_1)\delta(x_2-x'_2)$ being the two-dimensional δ-function, is replaced by the operator $A(q)$ with kernel

$$A(x,x';q) \equiv \mathrm{e}^{-q(x-x')}A(x,x') = \mathcal{A}(x,\partial_x+q)\delta(x-x'). \tag{2.8}$$

We refer to the operator $A(q)$ constructed in this way as the "extended" version of the differential operator $\mathcal{A}$. It is easy to see that in terms of the $(p,\mathbf{q})$-representation (2.1) the dependence on the $\mathbf{q}$-variables of the kernels of these extensions of the differential operators is polynomial. In particular, let D_j denote the extension of the differential operator ∂_{x_j} $(j=1,2)$, i.e. according to (2.8)

$$D_j(x,x';q) = (\partial_{x_j}+q_j)\delta(x-x'), \qquad j=1,2, \tag{2.9}$$

then D_j in the $(p,\mathbf{q})$-representation takes the form

$$D_j(p;\mathbf{q}) = -\mathrm{i}\mathbf{q}_j\delta(p). \tag{2.10}$$

An operator A can have an inverse in terms of the composition law (2.2), (2.3), say $AA^{-1}=I$ (in general left and right inverse can be different), where I is the unity operator,

$$I(x,x';q) = \delta(x-x'), \qquad I(p;\mathbf{q}) = \delta(p). \tag{2.11}$$

In order to make the inversion A^{-1} of an (extended) differential operator A uniquely defined we impose the condition that the product $(A^{-1})^{(\mathbf{s})}A^{-1}$ exists and is a bounded function of $\mathbf{s}$ in a neighborhood of $\mathbf{s}=0$. Let us consider as an example the operator $D_1 - a$, where, being a a complex constant, we write for shortness a instead of aI, according to a general notation we use in the following. For its inverse operator we get

$$(D_1-a)^{-1}(x,x';q) = \operatorname{sgn}(q_1-a_{\Re})\mathrm{e}^{(a-q_1)(x_1-x'_1)}\theta((q_1-a_{\Re})(x_1-x'_1))\delta(x_2-x'_2), \tag{2.12}$$

that is just the standard resolvent of the operator ∂_{x_1}. In terms of the $(p, \mathbf{q})$-representation this inverse operator is given by

$$(D_1 - a)^{-1}(p; \mathbf{q}) = \mathrm{i}(\mathbf{q}_1 - \mathrm{i}a)^{-1}\delta(p). \tag{2.13}$$

The simplicity of this formula makes clear the usefulness of the $(p, \mathbf{q})$-representation. Notice also that in this representation the necessity of requiring the boundedness condition introduced above for the inverse is especially evident since it is just this condition that excludes the presence of additional terms of the type $\delta(\mathbf{q}_1 - \mathrm{i}a)$ in (2.13).

According to this general construction the extended operator $L(q)$ corresponding to the operator (1.1) is given by

$$L = L_0 - U, \tag{2.14}$$

where $L_0 = -D_2 + D_1^2$, i.e. the extension of $\mathcal{L}(x, \partial_x)$ in the case of zero potential with kernels

$$L_0(x, x'; q) = \left[-\left(\partial_{x_2} + q_2\right) + \left(\partial_{x_1} + q_1\right)^2\right]\delta(x - x'), \quad L_0(p; \mathbf{q}) = (\mathrm{i}\mathbf{q}_2 - \mathbf{q}_1^2)\delta(p) \tag{2.15}$$

and U can be called the potential operator since has kernels

$$U(x, x'; q) = u(x)\delta(x - x'), \qquad U(p; \mathbf{q}) = v(p), \tag{2.16}$$

where $v(p) = (2\pi)^{-2}\int\mathrm{d}x\, \mathrm{e}^{\mathrm{i}px}u(x)$ is the Fourier transform of the potential $u(x)$. Below we always suppose that $u(x)$ is real, which by (2.4) is equivalent to

$$L^* = L. \tag{2.17}$$

The main object of our approach is the (extended) resolvent $M(q)$ of the operator $L(q)$, which is defined as the inverse of the operator L, that is

$$LM = ML = I. \tag{2.18}$$

In order to ensure the uniqueness of M we require that the product $M^{(\mathbf{s})}M$ is a bounded function in the neighborhood of $\mathbf{s} = 0$. Then, in particular, the resolvent M_0 of the bare operator L_0 has in the $(p, \mathbf{q})$-representation kernel

$$M_0(p; \mathbf{q}) = \frac{\delta(p)}{\mathrm{i}\mathbf{q}_2 - \mathbf{q}_1^2}. \tag{2.19}$$

In the x-representation we get

$$M_0(x, x'; q) = \frac{\mathrm{e}^{-q(x-x')}}{2\pi}\int\mathrm{d}\alpha\left[\theta(q_1^2 - q_2 - \alpha^2) - \theta(x_2 - x_2')\right]\mathrm{e}^{-\mathrm{i}\ell(\alpha+\mathrm{i}q_1)(x-x')}. \tag{2.20}$$

The resolvent of L can also be defined as the solution of the integral equations

$$M = M_0 + M_0UM, \qquad M = M_0 + MUM_0. \tag{2.21}$$

Under a small norm assumption for the potential we expect that the solution M is unique (the same for both integral equations) and that it satisfies the boundedness condition at $\mathbf{s} = 0$ for $M^{(\mathbf{s})}M$.

The resolvent is directly connected with the Green's functions of the operator (1.1). Indeed, since by definition (2.8) the product $\mathrm{e}^{q(x-x')}L(x, x'; q)$ is nothing but the kernel $L(x, x')$ of the original operator $\mathcal{L}(x, \partial_x)$ in (1.1), we have from (2.18)

$$\mathcal{L}(x, \partial_x)\big(\mathrm{e}^{q(x-x')}M(x, x'; q)\big) = \mathcal{L}^{\mathrm{d}}(x', \partial_{x'})\big(\mathrm{e}^{q(x-x')}M(x, x'; q)\big) = \delta(x - x'), \tag{2.22}$$

where $\mathcal{L}^{\mathrm{d}}$ is the operator dual to $\mathcal{L}$. Notice that while the product $\mathrm{e}^{q(x-x')}L(x, x'; q)$ is q-independent the same combination for the resolvent (see the simplest example in

(2.20)) essentially depends on q. This means that the resolvent can be considered as a two-parametric ($q \in \mathbb{R}^2$) family of Green's functions of the operator $\mathcal{L}$.

Thanks to (2.19) and (2.21) the kernel $M(p;\mathbf{q})$, like $M_0(p;\mathbf{q})$, is singular for $\mathbf{q} = \ell(\mathbf{q}_1)$ and $\mathbf{q}+p = \ell(\mathbf{q}_1 + p_1)$, where we introduced the two component vector

$$\ell(\alpha) = (\alpha, -\mathrm{i}\alpha^2), \tag{2.23}$$

such that $L_0(p;\ell(\mathbf{q}_1)) \equiv 0$. A special role in the theory is played by the operators ν and ω, whose kernels in the $(p,\mathbf{q})$-representation are given as values of $(ML_0)(p;\mathbf{q})$ and $(L_0M)(p;\mathbf{q})$, respectively, along these curves:

$$\nu(p;\mathbf{q}) = (ML_0)(p;\mathbf{q})\Big|_{\mathbf{q}=\ell(\mathbf{q}_1)}, \qquad \omega(p;\mathbf{q}) = (L_0M)(p;\mathbf{q})\Big|_{\mathbf{q}=\ell(\mathbf{q}_1+p_1)-p}. \tag{2.24}$$

It is clear by construction that both kernels $\nu(p;\mathbf{q})$ and $\omega(p;\mathbf{q})$ are independent of $\mathbf{q}_2$ and it is easy to see that both of them tend to $\delta(p)$ as $\mathbf{q}_1 \to \infty$. In the x-representation these kernels are given by means of the inversion of (2.1) as

$$\nu(x,x';\mathbf{q}_\Im) = \frac{\delta(x_2-x_2')}{2\pi}\int \mathrm{d}\mathbf{q}_{1\Re}\int \mathrm{d}x''\mathrm{e}^{\mathrm{i}\ell_\Re(\mathbf{q}_1)(x'-x'')}(ML_0)(x,x'';\ell_\Im(\mathbf{q}_1)), \tag{2.25}$$

$$\omega(x,x';\mathbf{q}_\Im) = \frac{\delta(x_2-x_2')}{2\pi}\int \mathrm{d}\mathbf{q}_{1\Re}\int \mathrm{d}x''\mathrm{e}^{\mathrm{i}\ell_\Re(\mathbf{q}_1)(x''-x)}\,(L_0M)(x'',x';\ell_\Im(\mathbf{q}_1)). \tag{2.26}$$

Operators ν and ω obey the conjugation properties

$$\nu^* = \nu, \qquad \omega^* = \omega, \tag{2.27}$$

which are equivalent to (2.17), and they are mutually inverse,

$$\omega\nu = I, \qquad \nu\omega = I. \tag{2.28}$$

The most essential property of these operators is that they dress M_0 and L_0 by the following formulae

$$M = \nu M_0 \omega, \qquad L = \nu L_0 \omega, \tag{2.29}$$

that thanks to (2.28) give

$$L\nu = \nu L_0, \qquad \omega L = L_0\omega, \qquad M\nu = \nu M_0, \qquad \omega M = M_0\omega. \tag{2.30}$$

Thus it is natural to call ν and ω dressing operators, or more specifically the Jost dressing operators, since the Jost solutions can be directly related to them by means of the following construction. Let us introduce

$$\Phi(x,\mathbf{k}) = \mathrm{e}^{-\mathrm{i}\ell(\mathbf{k})x}\chi(x,\mathbf{k}), \qquad \Psi(x,\mathbf{k}) = \mathrm{e}^{\mathrm{i}\ell(\mathbf{k})x}\xi(x,\mathbf{k}), \tag{2.31}$$

where

$$\chi(x,\mathbf{q}_1) = \int \mathrm{d}p\,\mathrm{e}^{-\mathrm{i}px}\nu(p;\mathbf{q}), \qquad \xi(x,\mathbf{q}_1) = \int \mathrm{d}p\,\mathrm{e}^{-\mathrm{i}px}\omega(p;\mathbf{q}-p) \tag{2.32}$$

and where we named $\mathbf{k}$ the spectral parameter $\mathbf{q}_1$ in order to meet the traditional notation. Then the first pair of equalities in (2.30) takes the form

$$\mathcal{L}(x,\partial_x)\Phi(x,\mathbf{k}) = 0, \qquad \mathcal{L}^{\mathrm{d}}(x,\partial_x)\Psi(x,\mathbf{k}) = 0 \tag{2.33}$$

so that Φ and Ψ solve the heat equation and its dual. In order to prove that they are the Jost solutions we first note that for χ and ξ in (2.32), thanks to (2.1) and (2.25), (2.26), we get

$$\chi(x,\mathbf{k}) = \int \mathrm{d}x'\,(ML_0)^{(\ell_\Re(\mathbf{k}))}(x,x';\ell_\Im(\mathbf{k})) = \int \mathrm{d}x'\,\nu^{(\mathbf{k})}(x,x';0), \tag{2.34}$$

$$\xi(x',\mathbf{k}) = \int \mathrm{d}x\,(L_0M)^{(\ell_\Re(\mathbf{k}))}(x,x';\ell_\Im(\mathbf{k})) = \int \mathrm{d}x\,\omega^{(\mathbf{k})}(x,x';0). \tag{2.35}$$

The first equalities here can be used in (2.24) and then we have by (2.31) and (2.32) that

$$\Phi(x,\mathbf{k}) = \int dx' \left(\mathcal{L}_0^{\mathrm{d}}(x',\partial_{x'})G(x,x',\mathbf{k})\right)e^{-i\ell(\mathbf{k})x'}, \tag{2.36}$$

$$\Psi(x',\mathbf{k}) = \int dx \left(\mathcal{L}_0(x,\partial_x)G(x,x',\mathbf{k})\right)e^{i\ell(\mathbf{k})x}, \tag{2.37}$$

where the Green's function of the Jost solution

$$G(x,x',\mathbf{k}) = e^{q(x-x')}M(x,x';q)\Big|_{q=\ell_{\Im}(\mathbf{k})} \tag{2.38}$$

appeared. Being a special case of the object considered in (2.22) it obeys

$$\mathcal{L}(x,\partial_x)G(x,x',\mathbf{k}) = \mathcal{L}^{\mathrm{d}}(x',\partial_{x'})G(x,x',\mathbf{k}) = \delta(x-x'). \tag{2.39}$$

From the integral equations (2.21) we have $ML_0 = I + M_0UML_0$ and $L_0M = I + L_0MUM_0$ that thanks to (2.36) and (2.37) give the integral equations

$$\Phi(x,\mathbf{k}) = e^{-i\ell(\mathbf{k})x} + \int dx'\, G_0(x-x',\mathbf{k})u(x')\Phi(x',\mathbf{k}), \tag{2.40}$$

$$\Psi(x,\mathbf{k}) = e^{i\ell(\mathbf{k})x} + \int dx'\, G_0(x'-x,\mathbf{k})u(x')\Psi(x',\mathbf{k}), \tag{2.41}$$

where the Green's function $G_0(x-x',\mathbf{k})$ is given by means of (2.38) in terms of the bare resolvent M_0. It is easy to see that thanks to (2.20)

$$G_0(x,\mathbf{k}) = \frac{1}{2\pi}\int d\alpha \left[\theta(\mathbf{k}_{\Re}^2 - \alpha^2) - \theta(x_2)\right] e^{-i\ell(\alpha+i\mathbf{k}_{\Im})x}, \tag{2.42}$$

which is the Green's function of the standard integral equations defining the Jost solutions. Self-conjugation property of the operators ν and ω (2.27) can be reformulated in terms of the Jost solutions $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$ in the following way

$$\overline{\Phi(x,\mathbf{k})} = \Phi(x,-\overline{\mathbf{k}}), \qquad \overline{\Psi(x,\mathbf{k})} = \Psi(x,-\overline{\mathbf{k}}) \tag{2.43}$$

as well as relations (2.28) that become the scalar product and completeness relation for the Jost solutions

$$\int dx_1 \Psi(x,\mathbf{k}+p)\Phi(x,\mathbf{k}) = 2\pi\delta(p), \quad p\in\mathbb{R}, \tag{2.44}$$

$$\int\limits_{x_2'=x_2} d\mathbf{k}_{\Re}\Psi(x',\mathbf{k})\Phi(x,\mathbf{k}) = 2\pi\delta(x_1-x_1'). \tag{2.45}$$

Correspondingly, we call the first and the second relation in (2.28) the scalar product and the completeness relation for the dressing operators.

The dressing operators themselves can be reconstructed by means of the Jost solutions,

$$\nu(x,x';q) = \frac{\delta(x_2-x_2')}{2\pi}e^{-q_1(x_1-x_1')}\int dk\; e^{i\ell(k+iq_1)x'}\Phi(x,k+iq_1), \tag{2.46}$$

$$\omega(x,x';q) = \frac{\delta(x_2-x_2')}{2\pi}e^{-q_1(x_1-x_1')}\int dk\; e^{-i\ell(k+iq_1)x}\Psi(x',k+iq_1). \tag{2.47}$$

They can be expanded into the formal series

$$\nu = I + \sum_{n=1}^{\infty}\nu_{-n}(2D_1)^{-n}, \qquad \omega = I + \sum_{n=1}^{\infty}(2D_1)^{-n}\omega_{-n}, \tag{2.48}$$

that in the $(p,\mathbf{q})$-representation have the meaning of asymptotic expansion in powers of $1/\mathbf{q}_1$ for $\nu(p;\mathbf{q})$ and of $1/(\mathbf{q}_1+p_1)$ for $\omega(p;\mathbf{q})$, i.e. the asymptotic expansions of the Jost solutions with respect to the spectral parameter at infinity. For the first coefficients in the expansion we get

$$\nu_{-1}+\omega_{-1}=0, \qquad [D_1,\nu_{-1}]=-[D_1,\omega_{-1}]=U, \tag{2.49}$$

that reconstruct the potential in the standard way, that is $u(x)=\partial_{x_1}\nu_{-1}(x)$.

In order to formulate the Inverse problem we introduce the operator

$$\rho=L_0ML_0-L_0. \tag{2.50}$$

that can be considered as a truncated resolvent. Then taking into account the definition of the derivative in (2.5) it is easy to show that the $\bar{\partial}$-equations for the dressing operators are given in the form

$$\bar{\partial}_1\nu=\nu R, \qquad \bar{\partial}_1\omega=-R\omega, \tag{2.51}$$

where the operator R is given by

$$R(p;\mathbf{q})=r(\mathbf{q}_1)\delta(p+2\ell_{\Re}(\mathbf{q}_1)), \tag{2.52}$$

with the spectral data $r(\mathbf{q}_1)$ defined by means of the following reduction of ρ:

$$r(\mathbf{q}_1)=2\pi\operatorname{sgn}(\mathbf{q}_{1\Re})\rho(p;\ell(\mathbf{q}_1))\Big|_{p=-2\ell_{\Re}(\mathbf{q}_1)}. \tag{2.53}$$

Let us mention that by construction the operator R obeys the conditions

$$[L_0,R]=0, \qquad [M_0,R]=0, \tag{2.54}$$

$$R^*=-R. \tag{2.55}$$

In terms of the Jost solutions equations (2.51) take the standard form

$$\overline{\partial}_{\mathbf{k}}\Phi(x,\mathbf{k})=\Phi(x,-\overline{\mathbf{k}})r(\mathbf{k}), \qquad \overline{\partial}_{\mathbf{k}}\Psi(x,\mathbf{k})=-\Psi(x,-\overline{\mathbf{k}})r(-\overline{\mathbf{k}}), \tag{2.56}$$

where $\overline{r(\mathbf{k})}=-r(-\overline{\mathbf{k}})$ (cf. (2.55)). In order to get the representation of the spectral data in terms of the Jost solutions we use (2.14) and (2.18) and rewrite (2.50) in the two equivalent forms $\rho=UML_0$ and $\rho=L_0MU$. Then for the value of ρ in (2.50) we get by (2.24), (2.31) and (2.32) that

$$r(\mathbf{k})=\frac{\operatorname{sgn}\mathbf{k}_{\Re}}{2\pi}\int\!\mathrm{d}x\,\mathrm{e}^{\mathrm{i}\ell(-\bar{\mathbf{k}})x}u(x)\Phi(x,\mathbf{k})=\frac{\operatorname{sgn}\mathbf{k}_{\Re}}{2\pi}\int\!\mathrm{d}x\,\mathrm{e}^{-\mathrm{i}\ell(\mathbf{k})x}u(x)\Psi(x,-\bar{\mathbf{k}}). \tag{2.57}$$

Finally, the normalization conditions that complete the formulation of the inverse problem for the Jost solution or its dual are given by (2.31) in the standard forms $\lim_{\mathbf{k}\to\infty}\chi(x,\mathbf{k})=1$ or, correspondingly, $\lim_{\mathbf{k}\to\infty}\xi(x,\mathbf{k})=1$ as follows from (2.32) and (2.48).

3. Similarity transformations of the Spectral Data

3.1. Rational similarity transformations

We study transformations of the above introduced objects generated by similarity transformations of the spectral data of the form

$$R'=WRW^{-1} \tag{3.1}$$

where the operator W is the following rational function of the operator D_1:

$$W=\prod_{j=1}^{N}\frac{D_1-a_j}{D_1-b_j}, \tag{3.2}$$

with a_j and b_j some complex parameters. Such W guarantees that R' is of the form (2.52) with

$$r'(\mathbf{k}) = \left(\prod_{j=1}^{N} \frac{\mathbf{k} - \mathrm{i}b_j}{\bar{\mathbf{k}} + \mathrm{i}b_j} \frac{\bar{\mathbf{k}} + \mathrm{i}a_j}{\mathbf{k} - \mathrm{i}a_j} \right) r(\mathbf{k}) \tag{3.3}$$

substituted for r. It is easy to see that such R' obeys properties (2.54), i.e. it commutes with L_0 and M_0, and in order for R' to obey property (2.55) we have to impose the following conditions on the parameters:

$$a_j = \bar{a}_{\pi_a(j)}, \qquad b_j = \bar{b}_{\pi_b(j)} \tag{3.4}$$

where π_a and π_b are some permutations of the indices. Notice that in the simplest situation where $N = 1$, i.e. $W = (D_1 - a_1)/(D_1 - b_1)$, the parameters must be real, $a_1 = \bar{a}_1$, $b_1 = \bar{b}_1$. Like in the nonstationary Schrödinger case the potential u' corresponding to this simplest situation can be obtained from the potential u by means of a binary Darboux transformation suggested in [22]. However, in contrast with the nonstationary Schrödinger equation, the generic case, with W given by (3.2) and a_j, b_j subjected to conditions (3.4) but not necessarily real, cannot be obtained by applying recursively binary Darboux transformations or one needs to admit non real intermediate potentials in the iterative procedure. Here we shall not use the techniques of the Darboux transformations but rather derive these transformations solving the inverse problem given by the spectral data R' explicitly in terms of the objects corresponding to the generic original spectral data R.

Let us mention that the new spectral data R' have additional discontinuities if compared with R. Indeed, $r'(\mathbf{k})$ given in (3.3) is undetermined at points $\mathbf{k} = \mathrm{i}a_j$ and $\mathbf{k} = \mathrm{i}b_j$ while $r(\mathbf{k})$ as given in (2.57) is a well defined function for $\mathbf{k}_{\Re} \neq 0$. Correspondingly, these additional discontinuities lead to new properties for the new dressing operators ν' and ω'. Indeed, we show below the solvability of the $\bar{\partial}$-equations

$$\bar{\partial}_1 \nu' = \nu' R' + \mathrm{i}\pi \sum_{j=1}^{N} \nu'_{a_j} \delta(D_1 - a_j), \qquad \bar{\partial}_1 \omega' = -R'\omega' + \mathrm{i}\pi \sum_{j=1}^{N} \delta(D_1 - b_j) \omega'_{b_j}, \tag{3.5}$$

that differ from equations (2.51) by additional δ-terms at $D_1 = a_j$ and at $D_1 = b_j$. In other words we show that ν' and ω' can have poles on the complex plane with residua ν'_{a_j} and ω'_{b_j}. Different choices of the form of these additional terms correspond to some renormalizations of the dressing operators and, consequently, of the Jost solutions*. One could also consider the case in which poles are absent, but we will show that, then, the solution does not contain solitons and decays at large distances.

In order to close the formulation of the inverse problem for ν' and ω' we have to impose normalization conditions on $\nu'(p; \mathbf{q})$ and $\omega'(p; \mathbf{q})$ at some value of $\mathbf{q}_1$ (both of them, as we know, must be independent on the variable $\mathbf{q}_2$). Taking into account that ν and ω are normalized by the asymptotic condition at infinity, it is natural to choose the same point for the ν' and ω'. Then it is easy to see that, without loss of generality (omitting the uninteresting case of a potential $u(x)$ shifted by a function of x_2 only), we can fix that

$$\nu'(p; \mathbf{q}) \to \delta(p), \qquad \omega'(p; \mathbf{q}) \to \delta(p), \quad \mathbf{q}_1 \to \infty. \tag{3.6}$$

* In [20], where the direct problem was examined, in order to define Jost solutions via an integral equation invariant in form, i.e. not depending on the parameters a_j and b_j, a different normalization was necessary. This is another intriguing characteristic of the heat equation in comparison with the nonstationary Schrödinger equation.

In this discussion we used that $\bar{\partial}_1(D_1 - a)^{-1} = \mathrm{i}\pi\delta(D_1 - a)$. The kernels of this δ-operator in the x- and $(p, \mathbf{q})$-representations are given by

$$\begin{aligned} \big(\delta(D_1 - a)\big)(p; \mathbf{q}) &= \delta(\mathbf{q}_1 - \mathrm{i}a)\delta(p), \\ \big(\delta(D_1 - a)\big)(x, x'; q) &= \frac{\mathrm{e}^{\mathrm{i}a_{\Im}(x_1 - x'_1)}}{2\pi}\delta(q_1 - a_{\Re})\delta(x_2 - x'_2), \end{aligned} \tag{3.7}$$

where we used the definition $\delta(z) = \delta(z_{\Re})\delta(z_{\Im})$ for any $z \in \mathbb{C}$. Note that in the $(p, \mathbf{q})$-representation $\nu'(p; \mathbf{q})$ and $\omega'(p; \mathbf{q})$ have poles, respectively, at $\mathbf{q}_1 = \mathrm{i}a_j$ and $\mathbf{q}_1 = \mathrm{i}b_j - p_1$ and we have

$$\nu'_{a_j}(p; \mathbf{q}) = -\mathrm{i} \operatorname*{res}_{\mathbf{q}_1 = \mathrm{i}a_j} \nu'(p; \mathbf{q}_1), \qquad \omega'_{b_j}(p; \mathbf{q}) = -\mathrm{i} \operatorname*{res}_{\mathbf{q}_1 = \mathrm{i}b_j - p_1} \omega'(p; \mathbf{q}_1). \tag{3.8}$$

Correspondingly, the Jost solutions $\Phi'(x, \mathbf{k})$ and $\Psi'(x, \mathbf{k})$ have poles, respectively, at $\mathbf{k} = \mathrm{i}a_j$ and $\mathbf{k} = \mathrm{i}b_j$ and in the x-representation we have

$$\nu'_{a_j}(x, x'; q) = \delta(x - x')\mathrm{e}^{\mathrm{i}\ell(\mathrm{i}a_j)x}\Phi'_{a_j}(x), \tag{3.9}$$

$$\omega'_{b_j}(x, x'; q) = \delta(x - x')\mathrm{e}^{-\mathrm{i}\ell(\mathrm{i}b_j)x'}\Psi'_{b_j}(x') \tag{3.10}$$

where

$$\Phi'_{a_j}(x) = -\mathrm{i} \operatorname*{res}_{\mathbf{k} = \mathrm{i}a_j} \Phi'(x, \mathbf{k}), \qquad \Psi'_{b_j}(x) = -\mathrm{i} \operatorname*{res}_{\mathbf{k} = \mathrm{i}b_j} \Psi'(x, \mathbf{k}). \tag{3.11}$$

Note that in the formulation of the inverse problem the values of the residua at poles ν'_{a_j} and ω'_{b_j} are left free. We show below that in order to close the construction of ν' and ω' it is necessary to impose that they dress the L_0 operator, that is, that they obey the differential equations like the first pair of equations in (2.30)

$$L'\nu' = \nu' L_0, \qquad \omega' L' = L_0 \omega', \tag{3.12}$$

where

$$L' = L_0 - U' \tag{3.13}$$

with some new potential u'. More precisely, we show in Section 3.2 that if ν' and ω' are dressing operators they satisfy the orthogonality relation $\omega'\nu' = I$ and, then, in Section 3.3, that this additional requirement imposed to ν' and ω' obtained from the solution of the $\overline{\partial}$ problem (3.5) is sufficient in order to guarantee that they satisfy the dressing equations (3.12). If poles are absent ν' and ω' are uniquely determined by (3.5) and their asymptotic behaviour at large $\mathbf{q}$ and it is not necessary to impose any additional requirement in order that they satisfy equations (3.12).

Let us note that the new dressing operators, since they obey (3.12), have the same (formal) expansions with respect to $\mathbf{q}_1$ as ν and ω in (2.48) with new ν'_{-n}, ω'_{-n} substituted for ν_{-n}, ω_{-n}.

We also assume that the new potential is real, i.e. $U'^* = U'$ and, correspondingly, $L'^* = L'$. Moreover, we assume that there exists the resolvent M' of the new operator L', that is its inverse operator. So we have also the second pair of equations in (2.30)

$$M'\nu' = \nu' M_0, \qquad \omega' M' = M_0 \omega'. \tag{3.14}$$

In order to solve the $\bar{\partial}$-equations for the new dressing operators in terms of the old ones it is convenient to consider, first, the $\bar{\partial}$-equations

$$\bar{\partial}_1(\nu' W \omega) = \nu'(\bar{\partial}_1 W)\omega, \qquad \bar{\partial}_1(\nu W^{-1}\omega') = \nu(\bar{\partial}_1 W^{-1})\omega', \tag{3.15}$$

that can be obtained from (2.51), (3.1) and (3.5) noting that W and W^{-1} cancel exactly the additional δ terms in (3.5). In order to integrate explicitly these equations we write first that by (3.2)

$$\bar{\partial}_1 W = \mathrm{i}\pi \sum_{j=1}^{N} c_j\, \delta(D_1 - b_j), \qquad \bar{\partial}_1 W^{-1} = \mathrm{i}\pi \sum_{j=1}^{N} \tilde{c}_j\, \delta(D_1 - a_j) \tag{3.16}$$

where

$$c_j = \frac{\prod_{l=1}^{N}(b_j - a_l)}{\prod_{l=1}^{N}{}'(b_j - b_l)}, \qquad \tilde{c}_j = \frac{\prod_{l=1}^{N}(a_j - b_l)}{\prod_{l=1}^{N}{}'(a_j - a_l)} \tag{3.17}$$

and $'$ means that the term $l = j$ is omitted. Thus (3.15) takes the form

$$\bar{\partial}_1(\nu' W \omega) = \mathrm{i}\pi \sum_{j=1}^{N} c_j \nu'_{b_j} \delta(D_1 - b_j)\omega_{b_j}, \tag{3.18}$$

$$\bar{\partial}_1(\nu W^{-1} \omega') = \mathrm{i}\pi \sum_{j=1}^{N} \tilde{c}_j \nu_{a_j} \delta(D_1 - a_j)\omega'_{a_j} \tag{3.19}$$

where new operators were introduced whose kernels in the $(p,\mathbf{q})$-representation are independent on $\mathbf{q}$ and are, precisely, values of the dressing operators at some points

$$\nu'_b(p;\mathbf{q}) = \nu'(p;\mathrm{i}b), \qquad \omega'_a(p;\mathbf{q}) = \omega'(p;\mathrm{i}a - p_1), \tag{3.20}$$

$$\nu_a(p;\mathbf{q}) = \nu(p;\mathrm{i}a), \qquad \omega_b(p;\mathbf{q}) = \omega(p;\mathrm{i}b - p_1). \tag{3.21}$$

Thanks to the composition law (2.3) and (3.7) it is easy to see that $\nu_a\delta(D_1 - a) = \nu\delta(D_1 - a)$, $\delta(D_1 - b)\omega_b = \delta(D_1 - b)\omega$, etc. Just these equalities were used in (3.18) and (3.19) where in the r.h.s.'s thanks to them only δ-functions have kernels that in $(p,\mathbf{q})$-representation depend on $\mathbf{q}_1$. This means that we can rewrite (3.18) and (3.19) as

$$\bar{\partial}_1 \left(\nu' W \omega - \sum_{j=1}^{N} \nu'_{b_j} \frac{c_j}{D_1 - b_j} \omega_{b_j} \right) = 0, \tag{3.22}$$

$$\bar{\partial}_1 \left(\nu W^{-1} \omega' - \sum_{j=1}^{N} \nu_{a_j} \frac{\tilde{c}_j}{D_1 - a_j} \omega'_{a_j} \right) = 0. \tag{3.23}$$

Since the kernels of ν', ν, ω', ω and W in the $(p,\mathbf{q})$-representation tend to $\delta(p)$ when $\mathbf{q}_1 \to \infty$, the expressions in parenthesis tend to I in the same limit and the equations can be explicitly integrated. Taking into account (2.28) we have

$$\nu' = \left(I + \sum_{j=1}^{N} \nu'_{b_j} \frac{c_j}{D_1 - b_j} \omega_{b_j} \right) \nu W^{-1}, \tag{3.24}$$

$$\omega' = W\omega \left(I + \sum_{j=1}^{N} \nu_{a_j} \frac{\tilde{c}_j}{D_1 - a_j} \omega'_{a_j} \right). \tag{3.25}$$

3.2. Scalar products of the dressing operators

Representations (3.24) and (3.25) obtained in the previous subsection are still undetermined, since they include the unknown multiplication operators ν'_{b_j} and ω'_{a_j}. In order to define them we need, first, to evaluate the scalar product of the dressing operators

$$S = \omega'\nu'. \tag{3.26}$$

Directly from this definition and thanks to (3.5) we have

$$\bar{\partial}_1 S = [S, R'] + \mathrm{i}\pi\sum_{j=1}^{N} S_{a_j}\delta(D_1 - a_j) + \mathrm{i}\pi\sum_{j=1}^{N}\delta(D_1 - b_j)S_{b_j}, \tag{3.27}$$

where we used notation (3.8) for the residuum at a pole. In getting, for instance, the second term in the r.h.s. we used the relation $\omega'\nu'_a\delta(D_1-a) = \omega'(\nu'(D_1-a))\delta(D_1-a) = (S(D_1-a))\delta(D_1-a) = S_a\delta(D_1-a)$, which is obtained by noting that ω' can be inserted into the bracket since it has no pole in a.

Thanks to (3.12) and (3.14) S must commute with L_0 and M_0. The condition of commutativity of S with M_0 formally is a consequence of the commutativity with L_0 as their are inverse one to another. However, as we already mentioned, the composition law (2.2) is not necessarly associative and therefore both conditions must be imposed. Let us consider $[L_0, S] = 0$ in the (p,$\mathbf{q}$)-representation. We obtain from (2.3) and (2.15) the equation $[\mathrm{i}p_2 - p_1(p_1 + 2\mathbf{q}_1)]\, S(p;\mathbf{q}) = 0$. Then $S(p;\mathbf{q})$ must be a generalized function concentrated at $p = 0$ and at $p + 2\ell_{\Re}(\mathbf{q}_1) = 0$ and, consequently, a linear combination with arbitrary coefficients depending on $\mathbf{q}_1$ of $\delta(p)$, $\delta(p + 2\ell_{\Re}(\mathbf{q}_1))$ and their derivatives up to a finite order. The derivative to be considered is $\left(\frac{\partial}{\partial p_1} - 2\mathrm{i}(p_1 + \mathbf{q}_1)\frac{\partial}{\partial p_2}\right)$ since it must annihilate $[\mathrm{i}p_2 - p_1(p_1 + 2\mathbf{q}_1)]$. This derivative when applied to $\delta(p + 2\ell_{\Re}(\mathbf{q}_1))$ can be equivalently written as $\frac{\partial}{\partial\bar{\mathbf{q}}_1}$ and, therefore, we conclude that the most general S commuting with L_0 has a kernel which is a finite linear combination (with coefficients depending on $\mathbf{q}_1$) of the following distributions

$$Y_n(p;\mathbf{q}) = \left[\frac{\partial}{\partial p_1} - 2\mathrm{i}(p_1 + \mathbf{q}_1)\frac{\partial}{\partial p_2}\right]^n \delta(p), \quad Z_n(p;\mathbf{q}) = \frac{\partial^n}{\partial\bar{\mathbf{q}}_1^n}\delta(p + 2\ell_{\Re}(\mathbf{q}_1)), \tag{3.28}$$

where $n = 0, 1, \ldots$. Let us consider now the condition $[M_0, S] = 0$. One can check that only Y_0 and Z_0 commute with M_0, while Y_n and Z_n for $n = 1, 2, \ldots$ do not commute with it and, precisely, the most singular terms in $[M_0, Y_n]$ and in $[M_0, Z_n]$ are, respectively, proportional to $\frac{\partial^{n-1}}{\partial\bar{\mathbf{q}}_1^{n-1}}\delta\left(\mathrm{i}\mathbf{q}_2 - \mathbf{q}_1^2\right)\delta(p)$ and $\frac{\partial^{n-1}}{\partial\bar{\mathbf{q}}_1^{n-1}}\delta\left(\mathrm{i}\mathbf{q}_2 - \mathbf{q}_1^2\right)\delta(p + 2\ell_{\Re}(\mathbf{q}_1))$, which, of course, does not contradict the statement that Y_n and Z_n commute with L_0. Therefore S can only be a linear combination of Y_0 and Z_0. But the term with Z_0 substituted in the l.h.s. of (3.27) produces a term Z_1 that cannot be compensated by a term in the r.h.s.

Thus finally we conclude that kernel of the operator S in the (p,$\mathbf{q}$)-representation must be of the form

$$S(p;\mathbf{q}) = s(\mathbf{q}_1)\delta(p). \tag{3.29}$$

Now turning back to the equation (3.27) we see that in the (p,$\mathbf{q}$)-representation the term in the l.h.s. as well as the second and third terms in the r.h.s. are proportional to

$\delta(p)$ while the first term in the r.h.s. is proportional to $\delta(p+2\ell_{\Re}(\mathbf{q}_1))$ due to (2.52). So this term has to be equal to zero:

$$[S,R'](p;\mathbf{q}) \equiv (s(-\bar{\mathbf{q}}_1)-s(\mathbf{q}_1))r'(\mathbf{q}_1)\delta(p+2\ell_{\Re}(\mathbf{q}_1))=0 \tag{3.30}$$

and (3.27) is reduced to

$$\bar{\partial}_1 S = \mathrm{i}\pi\sum_{j=1}^{N} S_{a_j}\delta(D_1-a_j)+\mathrm{i}\pi\sum_{j=1}^{N}\delta(D_1-b_j)S_{b_j}. \tag{3.31}$$

Taking into account that thanks to (3.6) and (3.26) at $\mathbf{q}_1$-infinity $S(p;\mathbf{q})\to\delta(p)$ we get

$$S = I+\sum_{j=1}^{N}\left(\frac{S_{a_j}}{D_1-a_j}+\frac{S_{b_j}}{D_1-b_j}\right) \tag{3.32}$$

or by (3.29)

$$s(\mathbf{q}_1)=1+\sum_{j=1}^{N}\left(\frac{\mathrm{i}S_{a_j}}{\mathbf{q}_1-\mathrm{i}a_j}+\frac{\mathrm{i}S_{b_j}}{\mathbf{q}_1-\mathrm{i}b_j}\right). \tag{3.33}$$

Finally, it is clear that for generic $r(\mathbf{q}_1)$ (and, thus, $r'(\mathbf{q}_1)$) equation (3.30) gives that $S_{a_j}=S_{b_j}=0$ for all j and thus we proved that

$$S=\omega'\nu'=I. \tag{3.34}$$

In other words we proved that like in the case of decaying potential the dressing operators obey the first equality in (2.28). It is necessary to mention that, on the contrary, as shown below, the second equality for the product $\nu'\omega'$ changes essentially.

3.3. Construction of Jost solutions and potential

In order to proceed with the construction of the dressing operators ν' and ω' it is convenient to work in the x-representation and to introduce the corresponding solutions $\Phi'(x,\mathbf{k})$ and $\Psi'(x,\mathbf{k})$ by means of the $'$ analog of the relations (2.31) and (2.32). Let us first consider the object $f_b=(D_1-b)^{-1}\omega_b\nu$ appearing in the expression for ν' in (3.24). By (2.12), (2.32) and (2.31) we have

$$f_b(x,x';q)=\frac{\delta(x_2-x_2')}{2\pi}\int d\alpha\, e^{\mathrm{i}(\ell(\alpha+\mathrm{i}q_1)-\ell(\mathrm{i}b))x+\mathrm{i}\alpha(x_1'-x_1)}\mathcal{F}(x,\mathrm{i}b,\alpha+\mathrm{i}q_1) \tag{3.35}$$

where the so called Cauchy–Baker–Akhiezer function introduced in [24]

$$\mathcal{F}(x,\mathbf{k},\mathbf{k}')=\int\limits_{\substack{(\mathbf{k}_{\Im}-\mathbf{k}'_{\Im})\infty\\ y_2=x_2}}^{x_1} dy_1\,\Psi(y,\mathbf{k})\Phi(y,\mathbf{k}') \tag{3.36}$$

had appeared. Then by using (2.34) we obtain

$$\Phi'(x,\mathbf{k})=\prod_{l=1}^{N}\frac{\mathbf{k}-\mathrm{i}b_l}{\mathbf{k}-\mathrm{i}a_l}\left(\Phi(x,\mathbf{k})+\sum_{j=1}^{N}c_j\Phi'(x,\mathrm{i}b_j)\mathcal{F}(x,\mathrm{i}b_j,\mathbf{k})\right). \tag{3.37}$$

Analogously from ω' in (3.25) we have

$$\Psi'(x,\mathbf{k})=\prod_{l=1}^{N}\frac{\mathbf{k}-\mathrm{i}a_l}{\mathbf{k}-\mathrm{i}b_l}\left(\Psi(x,\mathbf{k})-\sum_{j=1}^{N}\tilde{c}_j\mathcal{F}(x,\mathbf{k},\mathrm{i}a_j)\Psi'(x,\mathrm{i}a_j)\right). \tag{3.38}$$

The function $\mathcal{F}(x,\mathbf{k},\mathbf{k}')$ obeys the $\bar{\partial}$-equations

$$\frac{\partial}{\partial \bar{\mathbf{k}}'}\mathcal{F}(x,\mathbf{k},\mathbf{k}') = \mathrm{i}\pi\delta(\mathbf{k}-\mathbf{k}') + \mathcal{F}(x,\mathbf{k},-\bar{\mathbf{k}}')r(\mathbf{k}'), \tag{3.39}$$

$$\frac{\partial}{\partial \bar{\mathbf{k}}}\mathcal{F}(x,\mathbf{k},\mathbf{k}') = -\mathrm{i}\pi\delta(\mathbf{k}-\mathbf{k}') - \mathcal{F}(x,-\bar{\mathbf{k}},\mathbf{k}')r(-\bar{\mathbf{k}}), \tag{3.40}$$

that are often considered as a generalization of equations (2.56).

If $\Phi'(x,\mathbf{k})$ and $\Psi'(x,\mathbf{k})$ have no pole singularities, i.e. $\Phi'_{a_j}(x) \equiv \Psi'_{b_j}(x) \equiv 0$ (see (3.11)) for all j, then the denominators in the r.h.s.'s of (3.37) and (3.38) must be compensated by zeroes of the expressions in parentheses. These $2N$ equations uniquely determine the $2N$ functions $\Phi'(x,\mathrm{i}b_j)$ and $\Psi'(x,\mathrm{i}a_j)$ and then the Jost solutions $\Phi'(x,\mathbf{k})$ and $\Psi'(x,\mathbf{k})$ themselves. In this case it is easy to check that the corresponding dressing operators obey (3.12) that proves the statement given there for the absence of poles. Turning back to the generic situation, in order to complete the construction of $\Phi'(x,\mathbf{k})$ and $\Psi'(x,\mathbf{k})$ we evaluate the values $\Phi'(x,\mathrm{i}b_j)$ and $\Psi'(x,\mathrm{i}a_j)$ appearing in (3.37) and (3.38) by using the equality $\omega'\nu' = I$ derived in the previous section. Once constructed $\Phi'(x,\mathbf{k})$ and $\Psi'(x,\mathbf{k})$ in terms of $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$, by using the asymptotic limits $\lim_{x_2\to\infty}\chi(x,\mathbf{k}) = \lim_{x_2\to\infty}\xi(x,\mathbf{k}) = 1$, easily derived from (2.40) and (2.41), we must verify that $\chi'(x,\mathbf{k})$ and $\xi'(x,\mathbf{k})$ are polynomially bounded at space infinity and, then, that ν' and ω' belong to the considered space of operators, which guarantees the correctness of the above derived equalities and, in particular, of (3.34). We assume this behavior and below we shall prove it in some special cases.

It was already mentioned that the first equality in (2.28) can be rewritten in terms of Φ and Ψ as (2.44). The same orthogonality condition for Φ' and Ψ' follows from (3.34). Inserting in it (3.37) and (3.38) we get

$$\begin{aligned}\int \mathrm{d}x_1 &\left(\Psi(x,\mathbf{k}+p) - \sum_{j=1}^{N}\tilde{c}_j\mathcal{F}(x,\mathbf{k}+p,\mathrm{i}a_j)\Psi'(x,\mathrm{i}a_j)\right)\\ &\qquad\times\left(\Phi(x,\mathbf{k}) + \sum_{j=1}^{N}c_j\Phi'(x,\mathrm{i}b_j)\mathcal{F}(x,\mathrm{i}b_j,\mathbf{k})\right) = 2\pi\delta(p).\end{aligned} \tag{3.41}$$

Thanks to the above assumption it is clear that the integral exists in the sense of distributions. Now we multiply this equality by $\Psi(x',\mathbf{k})\Phi(x'',\mathbf{k}+p)$ where $x'_2 = x''_2 = x_2$, integrate it with respect to $\mathbf{k}_{\Re}$ and p and use (2.45). Then the r.h.s. becomes equal to $4\pi^2\delta(x'_1 - x''_1)$. Correspondingly, the l.h.s. at a generic point $\mathbf{k}$ must have zero $\partial_{\overline{\mathbf{k}}}$-derivative and the residua of poles at $\mathbf{k} = \mathrm{i}a_j$ and $\mathbf{k} = \mathrm{i}b_j$ must be equated to zero getting

$$\Psi'(x,\mathrm{i}a_j) = \sum_{l=1}^{N}\mathcal{F}'(x,\mathrm{i}a_j,\mathrm{i}b_l)c_l\Psi(x,\mathrm{i}b_l), \tag{3.42}$$

$$\Phi'(x,\mathrm{i}b_j) = -\sum_{l=1}^{N}\Phi(x,\mathrm{i}a_l)\tilde{c}_l\mathcal{F}'(x,\mathrm{i}a_l,\mathrm{i}b_j), \tag{3.43}$$

where in analogy with (3.36) we introduced

$$\mathcal{F}'(x,\mathbf{k},\mathbf{k}') = \int\limits_{\substack{(\mathbf{k}_{\Im}-\mathbf{k}'_{\Im})\infty\\ y_2=x_2}}^{x_1}\mathrm{d}y_1\,\Psi'(y,\mathbf{k})\Phi'(y,\mathbf{k}'). \tag{3.44}$$

Multiplying (3.42) by (3.43) we get

$$\Psi'(x, \mathrm{i}a_j)\Phi'(x, \mathrm{i}b_k) = -\sum_{l,m=1}^{N} \mathcal{F}'(x, \mathrm{i}a_j, \mathrm{i}b_l)c_l\Psi(x, \mathrm{i}b_l)\Phi(x, \mathrm{i}a_m)\tilde{c}_m\mathcal{F}'(x, \mathrm{i}a_m, \mathrm{i}b_k). \tag{3.45}$$

Now by (3.44) we rewrite this as

$$\frac{\partial}{\partial x_1}\mathcal{F}'(x, \mathrm{i}a_j, \mathrm{i}b_k) = -\sum_{l,m=1}^{N} \mathcal{F}'(x, \mathrm{i}a_j, \mathrm{i}b_l)c_l\left(\frac{\partial}{\partial x_1}\mathcal{F}(x, \mathrm{i}b_l, \mathrm{i}a_m)\right)\tilde{c}_m\mathcal{F}'(x, \mathrm{i}a_m, \mathrm{i}b_k), \tag{3.46}$$

or as

$$\frac{\partial}{\partial x_1}(\mathcal{F}')^{-1} = \frac{\partial}{\partial x_1}c\mathcal{F}\tilde{c}, \tag{3.47}$$

where we introduced matrices $\mathcal{F}$ and $\mathcal{F}'$ with elements

$$\mathcal{F}_{jk} = \mathcal{F}(x, \mathrm{i}b_j, \mathrm{i}a_k), \qquad \mathcal{F}'_{jk} = \mathcal{F}'(x, \mathrm{i}a_j, \mathrm{i}b_k) \tag{3.48}$$

and the two diagonal matrices $c = \mathrm{diag}\{c_1, \dots c_N\}$ and $\tilde{c} = \mathrm{diag}\{\tilde{c}_1, \dots \tilde{c}_N\}$. Let C denote an x_1-independent matrix. Thus we get

$$\mathcal{F}' = \tilde{c}^{-1}(C + \mathcal{F})^{-1}c^{-1} \tag{3.49}$$

and inserting $\mathcal{F}'$ in (3.42) and (3.43), correspondingly,

$$\Psi'(x, \mathrm{i}a_j) = \frac{1}{\tilde{c}_j}\sum_{l=1}^{N}(C + \mathcal{F})^{-1}_{jl}\Psi(x, \mathrm{i}b_l), \tag{3.50}$$

$$\Phi'(x, \mathrm{i}b_j) = \frac{-1}{c_j}\sum_{l=1}^{N}\Phi(x, \mathrm{i}a_l)(C + \mathcal{F})^{-1}_{lj}. \tag{3.51}$$

In order to avoid singularities we have to impose the condition that $\det(C + \mathcal{F}(x))$ has no zeroes on the x-plane. At the end of this Section we show that in the cases $N = 1$ and $N = 2$ this condition is formulated as a condition on admissible matrices C. For generic N we assume the existence of such matrices C that guarantee absence of zeroes of this determinant. In addition one has to check that for these C the matrix $\mathcal{F}'$ with elements $\mathcal{F}'_{jk}$ obtained from (3.44) by taking $\mathbf{k} = \mathrm{i}a_j$ and $\mathbf{k}' = \mathrm{i}b_k$ and by substituting, respectively, (3.50) and (3.51) for $\Phi'(x, \mathrm{i}b_k)$ and $\Psi'(x, \mathrm{i}a_j)$ coincides with (3.49). This leads to the equality

$$\mathcal{F}'_{jk}(x) = \int\limits_{(a_j - b_k)\infty}^{x_1} \mathrm{d}y_1 \frac{\partial}{\partial y_1}\mathcal{F}'_{jk}(y_1, x_2), \tag{3.52}$$

i.e. all elements of this matrix given by (3.49) have to obey conditions

$$\lim_{x_1 \to (a_j - b_k)\infty} \mathcal{F}'_{jk}(x) = 0. \tag{3.53}$$

It is easy to check these conditions in the cases $N = 1$ and $N = 2$, but for generic N we have to assume their validity.

Now from (3.37) and (3.38) we get for the residua of Φ' and Ψ'

$$\Phi'_{a_j}(x) = -\tilde{c}_j \sum_{l=1}^{N} \Phi'(x, \mathrm{i}b_l) c_l C_{lj}, \qquad \Psi'_{b_j}(x) = c_j \sum_{l=1}^{N} C_{jl} \tilde{c}_l \Psi'(x, \mathrm{i}a_l) \tag{3.54}$$

We can also insert the r.h.s.'s of (3.50) and (3.51) into (3.37) and (3.38) getting

$$\Phi'(x, \mathbf{k}) = \prod_{m=1}^{N} \frac{\mathbf{k} - \mathrm{i}b_m}{\mathbf{k} - \mathrm{i}a_m} \left(\Phi(x, \mathbf{k}) - \sum_{j,l=1}^{N} \Phi(x, \mathrm{i}a_j)(C + \mathcal{F})^{-1}_{jl} \mathcal{F}(x, \mathrm{i}b_l, \mathbf{k}) \right), \tag{3.55}$$

$$\Psi'(x, \mathbf{k}) = \prod_{m=1}^{N} \frac{\mathbf{k} - \mathrm{i}a_m}{\mathbf{k} - \mathrm{i}b_m} \left(\Psi(x, \mathbf{k}) - \sum_{j,l=1}^{N} \mathcal{F}(x, \mathbf{k}, \mathrm{i}a_j)(C + \mathcal{F})^{-1}_{jl} \Psi(x, \mathrm{i}b_l) \right). \tag{3.56}$$

The above formulas can be rewritten in terms of ratio of determinants as it was done for the KPI-case [18]. They can be considered as a special case of the construction given in [22] for generic eigenfunctions, i.e. not necessarily Jost solutions. On the contrary, here we deal with objects that indeed can be called Jost solutions as their departure from analyticity is under control and follows from (3.5) by the $'$ analog of the relations (2.31) and (2.32) and equalities (3.10):

$$\overline{\partial}_{\mathbf{k}} \Phi'(x, \mathbf{k}) = \Phi'(x, -\overline{\mathbf{k}}) r'(\mathbf{k}) + \mathrm{i}\pi \sum_{j=1}^{N} \Phi'_{a_j}(x) \delta(\mathbf{k} - \mathrm{i}a_j), \tag{3.57}$$

$$\overline{\partial}_{\mathbf{k}} \Psi'(x, \mathbf{k}) = -\Psi'(x, -\overline{\mathbf{k}}) r'(-\overline{\mathbf{k}}) + \mathrm{i}\pi \sum_{j=1}^{N} \Psi'_{b_j}(x) \delta(\mathbf{k} - \mathrm{i}b_j), \tag{3.58}$$

where $\Phi'_{a_j}(x)$ and $\Psi'_{b_j}(x)$ obey (3.54). Thanks to (3.11) and (3.55), (3.56) we get explicitly

$$\Phi'_{a_j}(x) = \tilde{c}_j \sum_{l,m=1}^{N} \Phi(x, \mathrm{i}a_l)(C + \mathcal{F})^{-1}_{lm} C_{mj}, \tag{3.59}$$

$$\Psi'_{b_j}(x) = c_j \sum_{l,m=1}^{N} C_{jl}(C + \mathcal{F})^{-1}_{lm} \Psi(x, \mathrm{i}b_m), \tag{3.60}$$

where we used notations (3.17) and (3.48). In particular, the above discussed case where $\Phi'(x, \mathbf{k})$ and $\Psi'(x, \mathbf{k})$ have no poles corresponds to matrix $C = 0$. These transformations do not add solitons and can be viewed as transformations of the continuous spectrum.

Now we can derive an explicit formula for the potential u' with spectral data (3.3). Inserting in (3.55) the two leading terms of the asymptotic expansion (2.48) and (2.48)$'$, i.e. $\chi(x, \mathbf{k}) = 1 - (2\mathrm{i}\mathbf{k})^{-1} \chi_{-1}(x) + \ldots$ and its analog for χ', we get thanks to (2.31)

$$\chi'_{-1}(x) = \chi_{-1}(x) + 2 \sum_{j=1}^{N} (a_j - b_j) - 2\partial_{x_1} \log \det(C + \mathcal{F}(x)), \tag{3.61}$$

so that for the potential we have

$$u'(x) = u(x) - 2\partial^2_{x_1} \log \det(C + \mathcal{F}(x)). \tag{3.62}$$

It is easy to check directly that if $\mathcal{L}'(x, \partial_x)$ denote the operator of the type (1.1) with potential $u'(x)$ substituted for $u(x)$ then (3.55) and (3.56) obey equations $\mathcal{L}'(x, \partial_x)\Phi'(x, \mathbf{k}) = 0$ and its dual if and only if

$$\partial_{x_2} C = 0. \tag{3.63}$$

Formula (3.62) is well known in the literature [21, 22], but nevertheless conditions that guarantee the regularity of the new potential are, to our knowledge, absent. This situation is completely different from the case of the Nonstationary Schrödinger Equation, where conditions of regularity were given in [18]. If all a_j and b_j in (3.2) are real, it is enough to choose the matrix C real in order to get a real potential. However, in the generic situation when (3.4) is satisfied, the conditions to impose to C in order to have a real potential are unknown. Even more complicated are the requirements to impose to the matrix C in order to obtain regular potentials.

We made a complete analysis in the cases $N = 1$ and $N = 2$ when the a_j and b_j are real. Then the matrix C must be real. In the case $N = 1$ the matrix C reduces to a real constant and the regularity condition is $(a_1 - b_1)C \geq 0$. The case $C = 0$ corresponds to a non solitonic situation and, in particular, when the original potential $u(x)$ is equal to zero we get that $u'(x) = 0$ also. In the case $N = 2$ a complete analysis is elementary but rather lengthy. Here, we give only the final result, more details are presented in the Appendix. It is convenient to introduce the following constants

$$D_{ij} = C_{ij} J_{i+1,j+1}, \qquad J_{ij} = \frac{1}{a_j - b_i}, \tag{3.64}$$

where i and j are defined $\bmod 2$ and to choose, for definiteness, $a_1 < a_2$ and $b_1 < b_2$.

In order to have a potential describing two solitons superimposed to a background at least two D_{ij} must be different from zero.

Then the potential is regular and has four rays if and only if for $i = 1, 2$

in the case $\det J < 0$**,** $J_{11}J_{22} > 0$

$$D_{ii} \leq 0, \qquad D_{i,i+1} > 0, \qquad \det C \leq 0 \tag{3.65}$$

in the case $\det J < 0$**,** $J_{11}J_{22} < 0$

$$D_{ii} < 0, \qquad D_{i,i+1} \geq 0, \qquad \det C \leq 0 \tag{3.66}$$

in the case $\det J > 0$

$$D_{ii} \geq 0, \qquad D_{i,i+1} < 0. \tag{3.67}$$

or

$$D_{ii} > 0, \qquad D_{i,i+1} \leq 0. \tag{3.68}$$

The potential is regular and has three rays if and only if for $i = 1$ or 2 and $j = 1$ or 2

in the case $\det J < 0$

$$D_{ii} = D_{j,j+1} = 0, \qquad D_{i+1,i+1} < 0, \qquad D_{j+1,j} > 0 \tag{3.69}$$

in the case $\det J > 0$

$$D_{ii} = D_{j,j+1} = 0, \qquad D_{i+1,i+1} > 0, \qquad D_{j+1,j} < 0. \tag{3.70}$$

If three D_{ij} are zero the potential contains only one soliton.

Notice that the two soliton solution $u'(x)$ can be considered as a surface in the (x, u') space depending on 8 parameters, i.e. D_{ij} $(i, j = 1, 2)$, a_i and b_i $(i = 1, 2)$. Since in the space of parameters the surfaces $D_{ij} = 0$ and $\det C = 0$ separate singular from regular solutions the soliton solution is not differentiable on these surfaces, which are therefore bifurcation surfaces according to the usual definition in catastrophe theory. Therefore, we expect that the solution, as a geometrical object, would be structurally instable at these values of the parameters.

The possible behaviours of the potential at large distances in the x-plane, according to the different choices of the parameters, are richer than in the KPI case. For instance, in the case $\det J > 0$, if we call $x_1 + hx_2 =$ const. the direction of a soliton, the directions at $x_2 = \pm\infty$ of the two solitons when the matrix C is full are given by $h = a_1 + a_2$ and $h = b_1 + b_2$, when the matrix C is diagonal by $h = a_1 + b_1$ and $h = a_2 + b_2$ and when the matrix C is off-diagonal by $h = a_1 + b_2$ and $h = a_2 + b_1$. In addition if only one element of the matrix C is zero the four rays of the solitons are directed along four different directions. These different behaviours are in agreement with the previous comment on the structural stability of the soliton solution. In the appendix we give a detailed description of the potentials in all regular cases.

4. Completeness relation and resolvent

We proved in section 3.2 that, like in the case of decaying potentials, the scalar product (3.34) $\omega'\nu'$ is equal to I or, in other words, that ν' is right inverse of ω'. On the contrary we expect, due to the presence of discrete data in the spectrum, that the second equality in (2.28), the so called completeness relation, is modified by an additional operator P' as follows

$$\nu'\omega' + P' = I. \tag{4.1}$$

From (3.34) we get directly that

$$P'\nu' = 0, \qquad \omega' P' = 0, \qquad P'^2 = P', \tag{4.2}$$

i.e. P' is an orthogonal projector. By substituting in $\nu'\omega'$ the values of ν' and ω' given in (3.24) and (3.25) we get an expression of P' in terms of the residua of ν' and ω'

$$P' = -\sum_{j=1}^{N} \nu'_{a_j} \frac{I}{D_1 - a_j} \omega'_{a_j} - \sum_{j=1}^{N} \nu'_{b_j} \frac{I}{D_1 - b_j} \omega'_{b_j} \tag{4.3}$$

that in the x-representation can be written explicitly as

$$\begin{aligned} P'(x, x'; q) = & -\operatorname{sgn}(x_1 - x'_1)\delta(x_2 - x'_2)\mathrm{e}^{-q_1(x_1 - x'_1)} \\ & \times \sum_{j=1}^{N} \{\theta((q_1 - b_{j\Re})(x_1 - x'_1))\Phi'(x, \mathrm{i}b_j)\Psi'_{b_j}(x') \\ & + \theta((q_1 - a_{j\Re})(x_1 - x'_1))\Phi'_{a_j}(x)\Psi'(x', \mathrm{i}a_j)\}. \end{aligned} \tag{4.4}$$

This equality can be simplified by means of (3.54)

$$\begin{aligned} P'(x, x'; q) = & -\delta(x_2 - x'_2)\mathrm{e}^{-q_1(x_1 - x'_1)} \\ & \times \sum_{j,l=1}^{N} \Phi'(x, \mathrm{i}b_j) c_j C_{jl} \tilde{c}_l \Psi'(x', \mathrm{i}a_l)\{\theta(q_1 - b_{j\Re}) - \theta(q_1 - a_{l\Re})\}. \end{aligned} \tag{4.5}$$

Then we see that P' is equal to zero outside the largest interval in q_1 with extremes $a_{l\Re}$, $b_{j\Re}$ $(j, l = 1, \ldots, N)$ and that $\overline{\partial} P'$ (see (2.6)) is equal to zero if q_1 is different from $a_{l\Re}$ and $b_{j\Re}$. Again it is necessary to emphasize that all these results are obtained on the basis of the assumption of polynomially boundedness of the χ' and ξ' formulated in the previous section.

Also the dressing formula for the resolvent M', inverse of L', must be corrected with respect to the bilinear representation (2.29) adding an operator m' as follows

$$M' = \nu' M_0 \omega' + m'. \tag{4.6}$$

Thanks to (3.12) we have $L'M' = L'\nu' M_0\omega' + L'm' = \nu'\omega' + L'm'$, where we used the fact that $L_0 M_0 = I$. Thus in order to obey $L'M' = I$ the operator m' by (4.1) has to obey the equality $L'm' = P'$.

Let us consider here the case $N = 1$ and let $a = a_1$, $b = b_1$ be real and let us label the corresponding quantities with 1. We have then

$$M_1 = \nu_1 M_0 \omega_1 + m_1 \tag{4.7}$$

and, in this case, it is easy to derive from (4.3) that

$$\begin{aligned} m_1 = \nu_{1,a} & \frac{1}{(D_1 - a)(D_2 - (a+b)D_1 + ab)} \omega_{1,a} \\ & + \nu_{1,b} \frac{1}{(D_1 - b)(D_2 - (a+b)D_1 + ab)} \omega_{1,b}. \end{aligned} \tag{4.8}$$

In writing the kernel of this operator in the x-representation we can use (3.54) that in this case thanks to (3.17) takes the form

$$\Phi_{1,a}(x) = c\Phi_1(x, \mathrm{i}b) \qquad \Psi_{1,b}(x) = -c\Psi_1(x, \mathrm{i}a), \tag{4.9}$$

where

$$c = (a-b)^2 C_{11}. \tag{4.10}$$

We get a formula similar to (4.5), that is

$$\begin{aligned} m_1(x, x'; q) = {} & \big(\theta(q_1 - a) - \theta(q_1 - b)\big)\big[\theta(x_2 - x_2') - \theta(-q_2 + (a+b)q_1 - ab)\big] \\ & \times c\mathrm{e}^{-q(x-x')}\Phi_1(x, ib)\Psi_1(x', \mathrm{i}a). \end{aligned} \tag{4.11}$$

From this equation we have $L_1 m_1 = P_1$ as required. Let us, however, stress that we would obtain this equality for any other x-independent second term in the square bracket and that the above specific choice is necessary in order to avoid an exponential growth of $m_1(x, x'; q)$ at space infinities, as easily follows from relations (3.50) and (3.51). For generic N the construction of m' and the study of its asymptotic properties are essentially more cumbersome.

Let us discuss here some properties of the resolvent M_1 constructed for the case $N = 1$ as they follow from (4.7) and (4.8)–(4.11). We see that, in comparison with the resolvent M (2.29) of a decaying potential, M_1 has an additional discontinuity at $q_2 = (a+b)q_1 - ab$ due to the term $m_1(x, x'; q)$. This discontinuity is not compensated by the first term in (4.7) and its presence in the resolvent M_1 is a characteristic manifestation of the solitonic content of the potential $u_1(x)$. The term $m_1(x, x'; q)$ is zero when q_1 is outside the interval (a, b) and therefore discontinuous also along the lines $q_1 = a$ and $q_1 = b$ on the q-plane. These discontinuities can be compensated by the pole behavior of the dressing operators ν_1 and ω_1 in the dressed term $\nu_1 M_0 \omega_1$ and need a detailed study.

From (3.5) using the Cauchy-Green formula and the asymptotic behaviour of ν_1 and ω_1 we have

$$\nu_1 = I - \frac{1}{\pi}\int \frac{\mathrm{d}^2\mathbf{z}_1}{\mathbf{z}_1}\left(\nu_1 R_1\right)^{(\mathbf{z}_1)} + \nu_{1,a}\frac{1}{D_1 - a}, \tag{4.12}$$

$$\omega_1 = I + \frac{1}{\pi}\int \frac{\mathrm{d}^2\mathbf{z}_1}{\mathbf{z}_1}\left(R_1\omega_1\right)^{(\mathbf{z}_1)} + \frac{1}{D_1 - b}\omega_{1,b} \tag{4.13}$$

where notation (2.5) for the shift was used. Then inserting them into $\nu_1 M_0 \omega_1$ we get

$$\nu_1 M_0 \omega_1 = M_0 - \frac{1}{\pi}\int \frac{\mathrm{d}^2\mathbf{z}_1}{\mathbf{z}_1}\nu_1^{(\mathbf{z}_1)}\left[R_1^{(\mathbf{z}_1)}, M_0\right]\omega_1^{(\mathbf{z}_1)} + M_{1,\mathrm{discr}}, \tag{4.14}$$

where

$$M_{1,\mathrm{discr}} = \nu_{1,a}\frac{M_0}{D_1 - a}\omega_{1,a} + \nu_{1,b}\frac{M_0}{D_1 - b}\omega_{1,b}. \tag{4.15}$$

In the x-representation we get for the contribution of the discrete part of the spectrum to $\nu_1 M_0 \omega_1$, thanks to (2.12), (2.20), (3.10), and (3.54),

$$M_{1,\mathrm{discr}}(x, x'; q) = c\Phi_1(x, \mathrm{i}b)\Psi_1(x', \mathrm{i}a)\mathrm{e}^{-q(x-x')}[\Gamma_a(x - x'; q) - \Gamma_b(x - x'; q)], \tag{4.16}$$

where

$$\Gamma_a(x; q) = \frac{\mathrm{e}^{\mathrm{i}\ell(\mathrm{i}a)x}}{2\pi}\int \mathrm{d}\alpha\left[\theta(q_1^2 - q_2 - \alpha^2) - \theta(x_2)\right]\frac{\mathrm{e}^{-\mathrm{i}\ell(\alpha+\mathrm{i}q_1)x}}{q_1 - a - \mathrm{i}\alpha}. \tag{4.17}$$

The singularities of Γ_a can be explicitly extracted getting

$$\Gamma_a(x; q) = \Gamma_{a,\mathrm{reg}}(x; q) + \Gamma_{a,\mathrm{sing}}(x; q), \tag{4.18}$$

where

$$\begin{aligned}\Gamma_{a,\mathrm{reg}}(x; q) = {} & \frac{\theta(x_2)}{2}\left\{1 - \operatorname{erf}\frac{x_1 + 2ax_2}{2\sqrt{x_2}}\right\} \\ & - \frac{\theta(q_1^2 - q_2)}{2\pi\mathrm{i}}\int\limits_{-\sqrt{q_1^2 - q_2}}^{\sqrt{q_1^2 - q_2}} \mathrm{d}\alpha\frac{\mathrm{e}^{-\mathrm{i}\ell(\alpha+\mathrm{i}q_1)x + \mathrm{i}\ell(\mathrm{i}a)x} - 1}{\alpha + \mathrm{i}(q_1 - a)}\end{aligned} \tag{4.19}$$

and

$$\Gamma_{a,\mathrm{sing}}(x; q) = -\theta(x_2)\theta(q_1 - a) + \frac{\theta(q_1^2 - q_2)}{\pi}\arctan\frac{\sqrt{q_1^2 - q_2}}{q_1 - a}. \tag{4.20}$$

We conclude that, for generic q_2, the discontinuities along the lines $q_1 = a$ and $q_1 = b$ of $M_{1,\mathrm{discr}}(x, x'; q)$ cancel exactly the discontinuities along the same lines of m_1, but $\mathrm{e}^{-qx}\Gamma_{a,\mathrm{sing}}(x; q)$ and $\mathrm{e}^{-qx}\Gamma_{b,\mathrm{sing}}(x; q)$ in the neighborhood, respectively, of the points (a, a^2) and (b, b^2) in the q-plane are ill defined due to the arctan.

One of the main advantages in using the resolvent is that the Green's functions can be obtained as specific reduction with respect to the parameter q of the resolvent. In particular, if we are interested in considering a smooth perturbation $u_2(x)$ (decaying at space infinity) of the potential $u_1(x)$ constructed above, then the Jost solution $\widetilde{\Phi}(x, \mathbf{k})$ of the perturbed potential $\tilde{u} = u_1 + u_2$ can be obtained as a perturbation of the Jost solution $\Phi_1(x, \mathbf{k})$ by means of the following integral equation

$$\widetilde{\Phi}(x, \mathbf{k}) = \Phi_1(x, \mathbf{k}) + \int \mathrm{d}x'\, G_1(x, x', \mathbf{k})u_2(x')\widetilde{\Phi}(x', \mathbf{k}), \tag{4.21}$$

where (cf. (2.38)) $G_1(x,x',\mathbf{k}) = \mathrm{e}^{q(x-x')}M_1(x,x';q)\big|_{q=\ell_{\Im}(\mathbf{k})}$. Taking into account that thanks to (2.23) $\ell_{\Im}(\mathbf{k}) = (\mathbf{k}_{\Im}, -\mathbf{k}_{\Re}^2 + \mathbf{k}_{\Im}^2)$ we get that the above mentioned discontinuity coming from m_1 becomes a discontinuity across the hyperbole $\left(\mathbf{k}_{\Im} - \frac{a+b}{2}\right)^2 - \mathbf{k}_{\Re}^2 = \left(\frac{a-b}{2}\right)^2$ which lies in the $\mathbf{k}$-plane outside the strip $a < \mathbf{k}_{\Im} < b$ if $a < b$ or $b < \mathbf{k}_{\Im} < a$ if $b < a$. On the other side the term in G_1 coming from m_1 is equal to zero outside this strip. Therefore, according to our previous discussion, only the discontinuities at $\mathbf{k} = \mathrm{i}a$ and at $\mathbf{k} = \mathrm{i}b$ are left. Thus in the case of the heat equation (at least for $N = 1$) the Green's function of the Jost solution has no an additional cuts in contrast with the case of the nonstationary Schrödinger equation [23, 15], but, anyway, due to the special singularities at $\mathbf{k} = \mathrm{i}a$ and at $\mathbf{k} = \mathrm{i}b$, the definition of the spectral data also for the case of a perturbed one soliton potential is not standard and needs a detailed analysis. This problem will be faced in a following work.

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Appendix

The potential $u'(x)$ describing two solitons superimposed to a generic background has at large distances in the x-plane a solitonic one dimensional behaviour along some rays. Any specific such ray can be represented by an equation $x_1 + hx_2 =$ const. with a given h and by specifying if the rays is pointing to $x_2 = +\infty$ or to $x_2 = -\infty$. We choose, for definiteness, $a_2 > a_1$ and $b_2 > b_1$.

Let us, first, list the cases in which the potentials has four rays.

It is convenient to rename the parameters a_i and b_i ($i = 1, 2$) as α_j ($j = 1, \ldots, 4$) in such a way that

$$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4. \tag{A.1}$$

If $C_{ij} \neq 0$ and $\det C \neq 0$ we have the following rays

h	versus
$\alpha_1 + \alpha_3$	$\pm\infty$
$\alpha_2 + \alpha_4$	$\pm\infty$

. (A.2)

Note that the rays are two by two parallel.

According to the remark we made at the end of Section 3 we expect that when an element of the matrix C or $\det C$ is taken to be zero the asymptotic behaviour at large distances will change discontinuously. If only one element of the matrix C or $\det C$ is zero we have four rays at large distances but they are no longer two by two parallel. If two elements not belonging to the same diagonal of C are zero we are left with only three rays. If three elements are zero the solution reduces to the one soliton solution and if the matrix C is zero the solution does not contain solitons.

The details are the following.

If, in the space of parameters of the soliton solution, we move to a bifurcation point by taking $C_{ii} = 0$ for $i = 1$ and/or 2 ($\det C \neq 0$) the four rays are obtained

by transforming discontinuously among the rays listed in the previous table (A.2) the following rays for $i = 1$ and/or 2

h	versus
$\alpha_1 + \alpha_3$	$(-)^{i+1} J_{11}\infty$
$\alpha_2 + \alpha_4$	$(-)^{i+1} J_{11}\infty$

(A.3)

into the rays

h	versus
$\alpha_1 + \alpha_4$	$(-)^{i+1} J_{11}\infty$
$\alpha_2 + \alpha_3$	$(-)^{i+1} J_{11}\infty$

. (A.4)

If $C_{i,i+1} = 0$ for $i = 1$ and/or 2 and $\det C \neq 0$ the four rays are obtained by substituting for $i = 1$ and/or 2 the following rays

h	versus
$\alpha_1 + \alpha_3$	$(-)^{i} \infty$
$\alpha_2 + \alpha_4$	$(-)^{i+1} \infty$

(A.5)

among the rays listed in the table (A.2) with the rays

h	versus
$\alpha_1 + \alpha_2$	$(-)^{i} \infty$
$\alpha_3 + \alpha_4$	$(-)^{i+1} \infty$

. (A.6)

If $\det C = 0$, $\det J < 0$ and $J_{11}J_{22} > 0$ and all $C_{ij} \neq 0$ the four rays are

h	versus
$\alpha_1 + \alpha_2$	$-J_{11}\infty$
$\alpha_1 + \alpha_3$	$+J_{11}\infty$
$\alpha_2 + \alpha_4$	$-J_{11}\infty$
$\alpha_3 + \alpha_4$	$+J_{11}\infty$

. (A.7)

If $\det C = 0$, $\det J < 0$ and $J_{11}J_{22} < 0$ and all $C_{ij} \neq 0$ the four rays are

h	versus
$\alpha_1 + \alpha_3$	$-J_{11}\infty$
$\alpha_1 + \alpha_4$	$+J_{11}\infty$
$\alpha_2 + \alpha_3$	$+J_{11}\infty$
$\alpha_2 + \alpha_4$	$-J_{11}\infty$

. (A.8)

Let us, now, list the cases in which the potential has three rays.

If $C_{ii} = 0$ and $C_{i,i+1} = 0$ for $i = 1$ or 2 the three rays are

h	versus
$a_1 + a_2$	$+J_{i+1,i+1}J_{i+1,i}\infty$
$a_i + b_{i+1}$	$(-)^{i+1} J_{i+1,i+1}\infty$
$a_{i+1} + b_{i+1}$	$(-)^{i} J_{i+1,i}\infty$

. (A.9)

If $C_{ii} = 0$ and $C_{i+1,i} = 0$ for $i = 1$ or 2 the three rays are

h	versus
$b_1 + b_2$	$-J_{i+1,i+1}J_{i,i+1}\infty$
$a_{i+1} + b_i$	$(-)^{i+1} J_{i+1,i+1}\infty$
$a_{i+1} + b_{i+1}$	$(-)^{i} J_{i,i+1}\infty$

. (A.10)

Finally, let us note that the corrections to this one dimensional solitonic behaviour are exponentially decaying, in contrast with the KPI case, where the corrections are rationally decaying at least in some regions of the plane [15].

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